In many disciplines (Science, Health Sciences, Business, Social Sciences ...) the relationships analysed are often not of only two variables but three or more. Often there is also some interplay between the ‘explanatory’ variables. Hence the response of the variable of interest to changes in one of the other variables, assuming the others are constant, is of interest. This is measured by **PARTIAL DIFFERENTIALS**.

**Examples:**

In **Business/ Marketing** the relationship between Price and Quantity is made much more realistic if consumer incomes, tastes and preferences, advertising and other variables are included in the analysis. An example is estimating the relationship between the number of passengers flown between Tahiti and Los Angeles (PPT-LAX) and airfares PPT-LAX (measured as yield in cents per km, a kind of weighted average of fares). The Quantity Demand -Price analysis is enhanced when other variables are added: GDPS USA, NZ and Australia (proxies for incomes), exchange rates (factors in prices), and other variables. A relationship equation with all those variables can be developed (with enough data, and if coefficients are of statistical significance). With a particular type of model, the **partial differentials** help provide a good estimate of the sensitivity of Quantity Demand to each of those variables.

They help answer questions like: Does dropping the price have more effect than incomes rising? Does advertising actually work? (PS That's Moorea in the distance, with an Air NZ plane landing at PPT. Heaven!)

**NOTATION**

Notation used in partial derivatives varies, depending on your teacher/ learning institution. The key thing to note is that partial notation is **different** from non-partial differentiation. As per:

The partial derivative of a function $f(x, y, \ldots)$ with respect to the variable $x$ is variously denoted by:

$$f_x', f_x, \frac{\partial f}{\partial x}, \text{ or } \frac{\partial f}{\partial x}.$$

Since in general a partial derivative is a function of the same arguments as was the original function, this functional dependence is sometimes explicitly included in the notation, as in:

$$f_x(x, y, \ldots), \frac{\partial f}{\partial x}(x, y, \ldots).$$

**METHOD OF PARTIAL DIFFERENTIATION BY EXAMPLE**

Say a particular variable (utility $U$) is the function of two other variables (your consumption of $X$ and of $Y$). And, based on revealed-preference market research, say that function is estimated as:

$$U = 7 \times X^5 \times Y^{\frac{1}{2}}$$

That is, utility is a product (multiplication) of three things: of 7, of the units of $X$ consumed (raised to the power of 5) and the units of $Y$ consumed (raised to the power of $\frac{1}{2}$, or the square root).

Recall: the square root of a number $Q$ is $Q^{\frac{1}{2}}$.

Which item of consumption, $X$ or $Y$, is most preferred? Let’s see.....
CALCULUS 07: PARTIAL DIFFERENTIALS

Looking at this Utility function, it would seem that you would get more by increasing X a little than by increasing Y a little. (Why?) Let’s prove this.

To measure the effect of each, we use **PARTIAL DIFFERENTIATION**.

**\( U = 7 \cdot X^5 \cdot Y^{\frac{1}{2}} \)**

Let’s do the partial differential of this Utility function **with respect to X**. So we treat **only the X as the variable**. The part of the function with the Y in it is treated **just as if it was a number**. Recall the differential of \( X^n \) is \( n \cdot X^{n-1} \). Here we will use the lower-case Greek delta notation, \( \delta \).

Hence

\[
\frac{\delta (U)}{\delta (X)} = 7 \cdot 5 \cdot X^{(5-1)} \cdot Y^{\frac{1}{2}}
\]

Simplifying this, we get

\[
\frac{\delta (U)}{\delta (X)} = 35 \cdot X^4 \cdot \sqrt{Y}
\]

This is the partial differential of U with respect to X.

From this, you will notice that a small change in X, no matter how Y changes, makes a large change in Utility U.

Now, let’s do the partial differential of the same Utility function with respect to Y; **it is the only variable** we consider. Now the part of the function with the X in it is treated just as if it was a number. Recall the differential of \( Y^n \) is \( n \cdot Y^{n-1} \).

Hence

\[
\frac{\delta (U)}{\delta (Y)} = 7 \cdot X^5 \cdot \frac{1}{2} \cdot Y^{(\frac{1}{2}-1)}
\]

Simplifying this, we get

\[
\frac{\delta (U)}{\delta (Y)} = 3.5 \cdot X^5 \cdot Y^{-\frac{1}{2}}
\]

Recall, negative powers are the inverse (“one over”) the positive power.

Simplifying this, we get

\[
\frac{\delta (U)}{\delta (Y)} = 3.5 \cdot X^5 \cdot \frac{1}{\sqrt{Y}}
\]

This is the partial differential of U with respect to Y.

From this you can estimate that a small change in Y (provided Y is larger than 1), no matter how X changes, makes **even less change in Utility U**.

**ACTIVITIES.** The following are recommended:

- Have a look at this Mr Patrick video: [https://www.youtube.com/watch?v=SbfRDBmyAMI](https://www.youtube.com/watch?v=SbfRDBmyAMI)


- If you would like to go a little further, this video on PDE’s (partial differential equations) is interesting, and the narrator has a calm, clean style: [https://www.youtube.com/watch?v=LYslBqjQTDI](https://www.youtube.com/watch?v=LYslBqjQTDI)